

# Einstein's mirror revisited

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I describe a simple geometrical derivation of the formula for reflection of light from a uniformly moving plane mirror directly from the postulates of special relativity.

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## I. INTRODUCTION

Reflection of light from a plane mirror in uniform rectilinear motion is a century-old problem, intimately related to the foundations of special relativity.<sup>1,2,3,4,5,6,7,8,9,10,11</sup> The reflection formula is usually obtained by referring to the Lorentz transformation to switch from the mirror's rest frame to the frame where the mirror moves at a constant velocity.<sup>4,12,13,14,15,16</sup> The reflection formula also follows from the constancy of the speed of light, by using Huygens' construction<sup>17</sup> or Fermat's principle of least time<sup>18</sup>.

In this paper, I introduce a derivation of the reflection formula by comparing the geometry of the problem in the frame where the mirror is moving to the one in the frame where the mirror is stationary. I analyze three distinct cases of the mirror's motion, when the mirror is moving: a) parallel to its surface; b) perpendicular to its surface; and c) in an arbitrary direction. The derivation requires nothing but a simple plane geometry and an elementary understanding of the postulates of special relativity, and does not use Lorentz transformation or any additional tools of classical wave optics.

## II. REFLECTION FROM A PLANE MIRROR MOVING UNIFORMLY PARALLEL TO ITS SURFACE

A plane mirror of length  $2\ell = \overline{LR}$  is placed horizontally in its rest frame (see Fig. 1). A photon is emitted from a source  $A$  located at a vertical distance  $h$  from the left edge  $L$  of the mirror. The photon bounces off the midpoint  $O$  from the mirror, and it is absorbed by the detector  $B$  located at the same vertical distance  $h$  from the right edge  $R$  of the mirror. Evidently, the angle of incidence and the angle of reflection of the photon are equal.

We now transfer to the frame in which the mirror, the source and the detector are all moving at a constant speed  $v$  to the right (see Fig. 2). Since the point of reflection in the mirror's rest frame divides its length into two equal halves, it must also divide the mirror in two equal halves with respect to any other inertial frame.<sup>19</sup> By making this claim, we implicitly invoked the principle of relativity, according to which one cannot distinguish one inertial frame from another. Also, at each instant of time, the

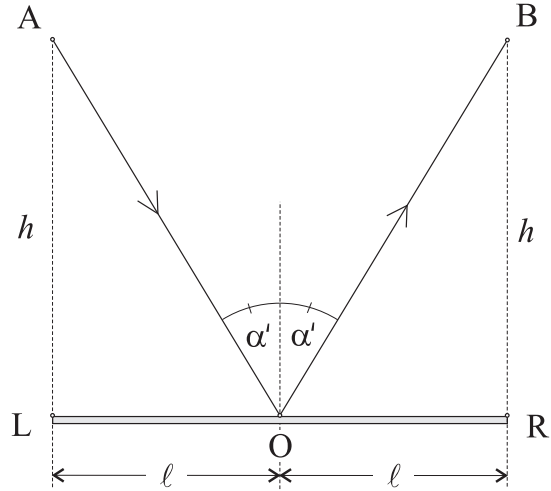


FIG. 1: Reflection of the photon from a horizontal mirror with respect to the mirror's rest frame.

moving source and the moving detector are located at a vertical distance  $h$  above the left and the right edges of the moving mirror, respectively, as they were in the frame where the mirror was stationary. Here, we again appealed to the principle of relativity which directly implies the invariance of the lengths measured perpendicularly to the direction of relative motion between the frames.<sup>20</sup>

In Fig. 2 we see that the photon was emitted from the moving source at the time when the source was located at  $A_1$ . At this instant, the left edge of the mirror was located at point  $L_1$  at a vertical distance  $h$  below  $A_1$ , and  $O_1$  is the location of the midpoint of the mirror. The photon bounces off the midpoint  $O_2$  a time  $t_1$  from its emission at  $A_1$ . During this time  $t_1$ , the photon has traveled the path  $\overline{A_1O_2} = ct_1$ , and the mirror, the source and the detector have moved the distance  $vt_1$  to the right. We have used Einstein's second postulate that the photon will move at a speed of light  $c$  with respect to every inertial frame of reference.

During the time  $t_2$  from the reflection, the photon has traversed the path  $\overline{O_2B_3} = ct_2$  being absorbed by the detector at point  $B_3$ . Accordingly, the mirror, the source and the detector have moved the additional distance  $vt_2$  to the right. At the time of the absorption, the right edge of the mirror was located at point  $R_3$  at a distance  $h$  vertically below  $B_3$ , and  $O_3$  is the location of the midpoint

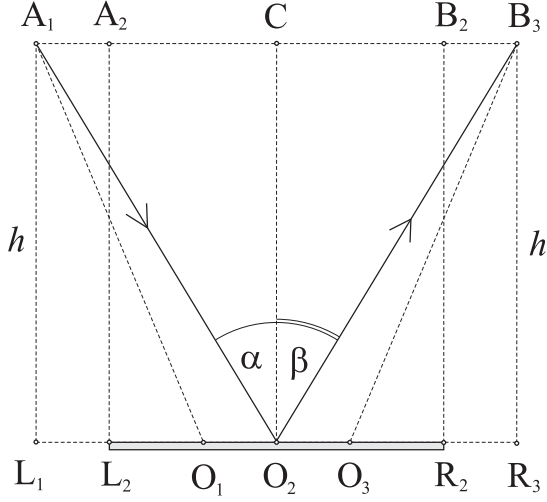


FIG. 2: The situation in Fig. 1 with respect to an observer moving at a constant speed  $v$  to the left.

of the mirror. If we denote by  $\alpha$  the angle of incidence, and by  $\beta$  the angle of reflection of the photon, and note the triangle similarities in Fig. 2, we may write:

$$\begin{aligned}\alpha &= \angle A_1 O_2 C = \angle L_1 A_1 O_2 \\ &= \angle L_1 A_1 O_1 + \angle O_1 A_1 O_2,\end{aligned}\quad (1)$$

$$\begin{aligned}\beta &= \angle C O_2 B_3 = \angle O_2 B_3 R_3 \\ &= \angle O_2 B_3 O_3 + \angle O_3 B_3 R_3.\end{aligned}\quad (2)$$

Obviously,  $\triangle A_1 L_1 O_1 = \triangle B_3 R_3 O_3$ , and thus:

$$\angle O_3 B_3 R_3 = \angle L_1 A_1 O_1. \quad (3)$$

Applying the sine theorem to  $\triangle O_1 A_1 O_2$ , we obtain:

$$\sin \angle O_1 A_1 O_2 = \frac{\overline{O_1 O_2}}{\overline{A_1 O_2}} \sin \angle A_1 O_1 O_2. \quad (4)$$

Since  $\overline{O_1 O_2} = vt_1$  and  $\overline{A_1 O_2} = ct_1$ , we get:

$$\sin \angle O_1 A_1 O_2 = \frac{v}{c} \sin \angle A_1 O_1 O_2. \quad (5)$$

Analogously, for  $\triangle O_2 B_3 O_3$  we obtain:

$$\sin \angle O_2 B_3 O_3 = \frac{v}{c} \sin \angle O_2 O_3 B_3, \quad (6)$$

where we have taken into account that  $\overline{O_2 O_3} = vt_2$  and  $\overline{O_2 B_3} = ct_2$ . But  $\angle A_1 O_1 O_2 = \angle O_2 O_3 B_3$ , which we use into Eqs. (5) and (6) to get:

$$\angle O_2 B_3 O_3 = \angle O_1 A_1 O_2. \quad (7)$$

From Eqs. (1), (2), (3) and (7) we deduce  $\alpha = \beta$ , that is, the incident angle of the photon equals the reflected angle. Thus, the motion of the mirror in its plane will not affect the reflection of the photon, and the photon will be reflected in accordance with the usual law of reflection as if the mirror were stationary.

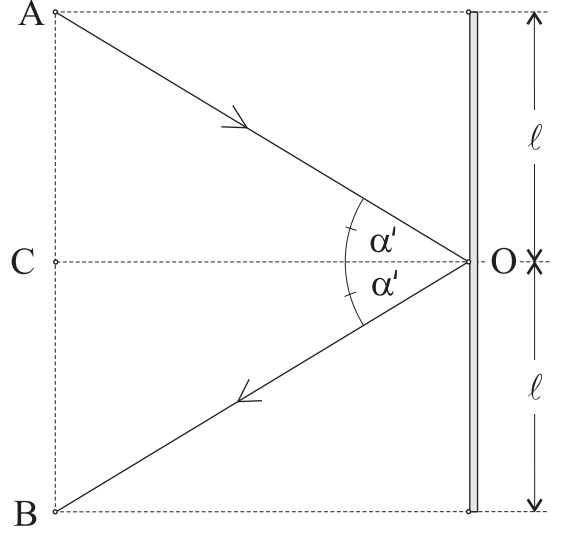


FIG. 3: Reflection at a vertical mirror observed from a frame where the mirror is stationary.

### III. REFLECTION FROM A PLANE MIRROR MOVING UNIFORMLY PERPENDICULAR TO ITS SURFACE

In this section we consider the reflection of the photon when the velocity of the mirror is normal to its surface. The schematic of the reflection with respect to the mirror's rest frame is shown in Fig. 3. The photon is emitted from a fixed source at  $A$ , bounces off the mirror at its midpoint  $O$ , and eventually hits the fixed detector at  $B$ . The source  $A$  and the detector  $B$  are located at a vertical line  $ACB$  parallel to the mirror's surface, being equally distant, but at the opposite sides, from the line  $OC$  normal to the mirror's surface ( $\overline{AC} = \overline{BC} = \ell$ ). The source  $A$  is horizontally aligned with the upper edge of the mirror, and the detector  $B$  with its lower edge. A simple trig reveals that the photon will be reflected off the mirror at the same angle at which it was incident.

The situation with respect to the frame where the mirror, the source and the detector are moving at a constant speed  $v$  to the right is shown in Fig. 4. We will apply the same arguments as in the previous section to derive the formula for reflection of the photon in this case.

After being emitted from the moving source at time when the source was located at  $A_1$ , the photon is reflected from the moving mirror off its midpoint  $O_2$  and then absorbed by the moving detector at the point  $B_3$ . If  $t_1$  is the photon's transit time from  $A_1$  to  $O_2$ , and  $t_2$  the time from  $O_2$  to  $B_3$ , we have  $\overline{A_1 O_2} = ct_1$  and  $\overline{O_2 B_3} = ct_2$ , where we have taken into account the constant speed of light postulate. At time when the photon reaches the detector at  $B_3$ , the detector has moved the distance  $\overline{B_1 B_3} = v(t_1 + t_2)$  to the right from its position  $B_1$  when the photon was emitted from  $A_1$ . From

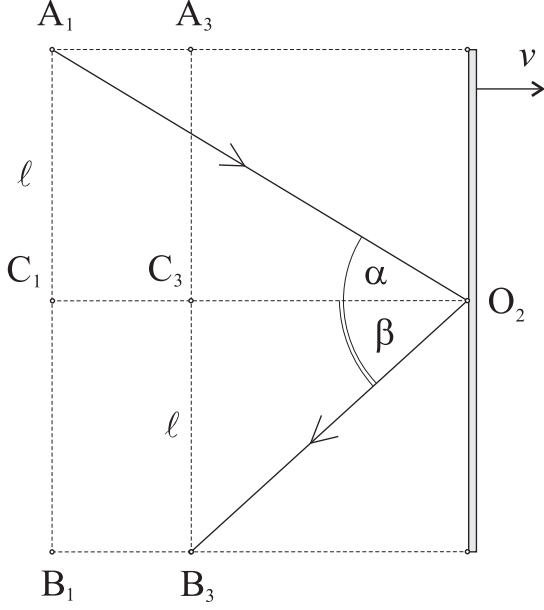


FIG. 4: The situation in Fig. 3 observed from a reference frame moving at a constant speed  $v$  to the left.

$\triangle A_1 C_1 O_2$  and  $\triangle B_3 C_3 O_2$  in Fig. 4, we have:

$$\cos \alpha = \frac{b}{ct_1}, \quad (8)$$

$$\cos \beta = \frac{b - v(t_1 + t_2)}{ct_2}, \quad (9)$$

where  $\alpha$  and  $\beta$  are the incident angle and the reflected angle of the photon, respectively, and  $b = \overline{C_1 O_2}$ . We eliminate  $b$  from Eqs. (8) and (9) to get:

$$\cos \beta = \left( \cos \alpha - \frac{v}{c} \right) \frac{t_1}{t_2} - \frac{v}{c}. \quad (10)$$

We apply the Pythagoras theorem to  $\triangle A_1 C_1 O_2$  and  $\triangle B_3 C_3 O_2$  to obtain:

$$(ct_1)^2 = \ell^2 + b^2, \quad (11)$$

$$(ct_2)^2 = \ell^2 + [b - v(t_1 + t_2)]^2, \quad (12)$$

where  $\ell = \overline{A_1 C_1} = \overline{B_3 C_3}$ . By subtracting Eq. (12) from Eq. (11) to eliminate  $\ell$ , we obtain:

$$(ct_1)^2 - (ct_2)^2 = b^2 - [b - v(t_1 + t_2)]^2. \quad (13)$$

The last equation can be factored into the form:

$$c^2(t_1 - t_2)(t_1 + t_2) = v(t_1 + t_2)[2b - v(t_1 + t_2)], \quad (14)$$

which upon division by  $(t_1 + t_2)$  and using Eq. (8) can be recasted into:

$$\frac{t_1}{t_2} = \frac{1 - v^2/c^2}{1 - 2(v/c) \cos \alpha + v^2/c^2}. \quad (15)$$

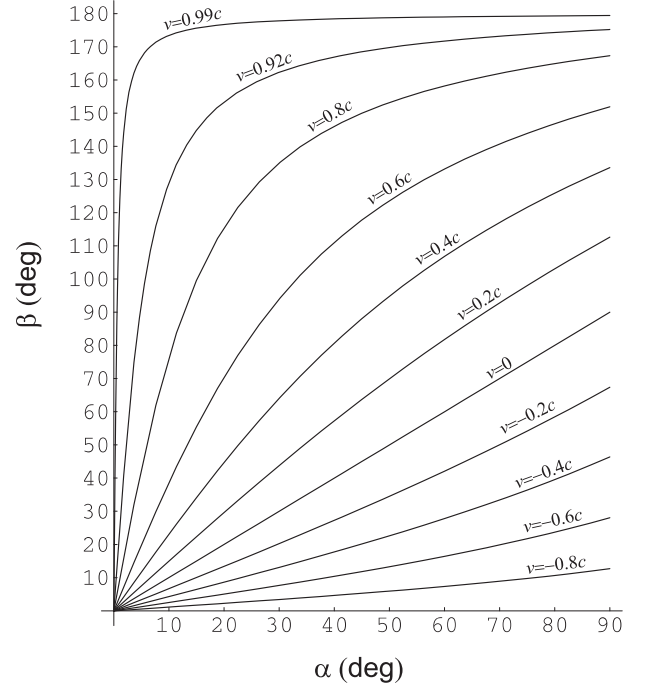


FIG. 5: The reflection angle  $\beta$  versus the incident angle  $\alpha$  of the photon for different values of the speed  $v$  of the mirror moving perpendicularly to its surface.

We substitute the last expression for  $t_1/t_2$  into Eq. (10) to obtain:

$$\cos \beta = \frac{-2v/c + (1 + v^2/c^2) \cos \alpha}{1 - 2(v/c) \cos \alpha + v^2/c^2}. \quad (16)$$

The result gives the angle of reflection  $\beta$  of the photon in terms of the incident angle  $\alpha$  and the speed  $v$  of the mirror when the mirror is moving perpendicularly to its surface. It exactly matches the reflection formula obtained otherwise.<sup>4,12,16,17,18</sup>

The behavior of the reflected angle  $\beta$  as a function of  $\alpha$  for different values of the speed  $v$  of the mirror is given in Fig. 5. Note that the negative values of  $v$  correspond to the motion of the mirror in the opposite direction to the one given in Fig. 4. From Fig. 5 it is evident that the reflected photon no longer obeys the usual law of reflection, except when  $v = 0$  which is the case when the mirror is stationary. It is also evident that  $\beta < \alpha$  when  $v < 0$ , and  $\beta > \alpha$  when  $v > 0$ .

An interesting property of the formula (16) is observed for the case  $v > 0$ . Namely, for a given positive value for the speed  $v$  of the mirror, there exist an interval of values for the incident angle  $\alpha$  of the photon ( $\alpha_c < \alpha < 90^\circ$ ), for which the photon is reflected at angles larger than  $90^\circ$ . Hence, the photon, instead of moving away from the mirror, moves in the same general direction as the mirror. The critical angle  $\alpha_c$  can be calculated from the formula:

$$\cos \alpha_c = \frac{2v/c}{1 + v^2/c^2}, \quad (17)$$

which follows from Eq. (16) by applying the condition  $\beta = 90^\circ$ . The effect is known as "the forward reflection of light".<sup>21</sup>

#### IV. REFLECTION FROM A PLANE MIRROR MOVING IN AN ARBITRARY DIRECTION AT A CONSTANT VELOCITY

We may generalize the above approach to the case when the reflection occurs off the mirror moving at a speed  $v$  directed at an angle  $\varphi$  from its surface normal. Since the motion of the mirror in its plane does not affect the reflection (see Sec. II), the reflection formula in

this case follows from Eq. (16) in Sec. III if we simply replace  $v$  by  $v \cos \varphi$ , which is the velocity component of the mirror that is normal to its surface:

$$\cos \beta = \frac{-2(v/c) \cos \varphi + [1 + (v^2/c^2) \cos^2 \varphi] \cos \alpha}{1 - 2(v/c) \cos \alpha \cos \varphi + (v^2/c^2) \cos^2 \varphi}. \quad (18)$$

When  $\varphi = 0$ , the motion of the mirror is perpendicular to its surface, and Eq. (18) reduces to Eq. (16). Also, when  $\varphi = 90^\circ$ , the mirror moves parallel to its surface, and the reflection formula simplifies to  $\alpha = \beta$ .

The detailed analysis of Eq. (18) for different values of  $\varphi$  is left to the student as an exercise.

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